Concerning the Calculation of Higher Derivative Complex Functions

by G. N. Duboshin

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In ~~the~~ analytical celestial mechanics as in other scientific disciplines, there is often a requirement to find a way of solving differential equation systems, ~~This also applies to the~~ as well as finite equation systems in the form of a series arranged by whole ~~studied in an order of~~ positive ~~degrees~~ with, either one or several parameters entering into these ~~questions~~ equations.

Usually, from the very appearance of the equations and implementation of theories in existence, it is possible to present the solution of a similar series of equations which interest us, so that the problem of the computation ~~is a matter of~~ results in determining the coefficients of these series.

These coefficients, as it is well-known, are determined successively in ~~number~~ order of their numerical increase either by single-type systems of linear, non-uniform equations (provided the solution ~~is of~~ involves finite ~~equations~~ equation system) or by single-type systems of linear, differential, non-uniform equations if the solution ~~is~~ involves of a system of differential equations.

Both ~~of~~ these and other systems are usually obtained as a result of substituting into the initial equation, instead of unknown functions, determining their above-mentioned series and following this with a comparison of coefficients with the same parameter degrees in the left and right portions of the equations.

These operations do not present any principal difficulties, but usually require the fulfillment of cumbersome and tiresome calculations, since the left sides portion of finite equations, which must be solved, or the right side of differential equations which must be integrated, in themselves represent an appearance of series situated in whole degrees of increase of unknown functions and parameters.

In this manner, it is necessary to substitute series for series and then perform computations and reductions.

These computations are slightly facilitating ~~decreased in complexity~~ ~~of~~ if the equations systems which serve to ~~for the determination of~~ unknown coefficients of the desired series are deducted by way of subsequent differentiation of initial equations by parameters. The calculating of unknown function quantities of these parameters is then followed by substitution in the resultant equations entering into their parameters of zero value. This manner of obtaining equations that determine the coefficients of the desired series leads to the computation of sequential derivatives from complex functions with several intermittent arguments amplitude and demands the implementation of general formulas for higher derivative complex functions.

Even though similar formulas were examined on numerous occasions and were deduced by many researchers mainly for the simplest function cases from one variable with one intermittent argument amplitude, they were essentially of little use, since their utilization requires lengthy computations and often it is easier to fulfill several sequential differentiations than to compile necessary higher derivatives with the aid of general formulas.

However, by ~~examining~~ the previously mentioned general formulas, certain useful recurrent relationships may be deduced between certain functions having operational characteristics; such ~~which~~ relationships permit rather rapidly and without much effort, compilation of expressions for highest derivatives not utilizing general formulas.

Since similar recurrent relationships in the available literature were ~~was~~ not discovered, then keeping in mind the expediency and a decrease in difficulty for the persons computing these problems, we decided to publish an article concerning this relatively elementary question. The purpose of this publication was to point out certain simple formulas, compiling these with explanations of a practical character and to give certain examples of their utilization.

We note in our conclusion, that this work examines only continuous ~~uninterrupted~~ functions, differentiated as many times as necessary by any of the arguments ~~items~~ entering into them ~~into the amplitudes~~.

I.

Let the function $z = f(x)$ be given, uninterrupted and differentiated any number of times by an independent variable x. If x has a certain continuous function of parameter t, also differentiated any number of times, then z will be z complex function from t, the first derivative of which is calculated by a well known formula

$$\frac{dz}{dt} = \frac{df(x)}{dx} \cdot \frac{dx}{dt}.$$

Utilizing the sequential rule given by this formula, the derivative of any series from z by t may be found. It is also possible to find a general formula which permits computation of any derivative while not producing all intermediate differentiations which become more cumbersome and tiring with an increase in numbers.

Similar general formulas, of which Leibonitz is the simplest example, for the derivatives of any series from the production of 2 functions were on numerous occasions, deduced by different methods by many prominent mathematicians and presented in many ways. By an account of N. Ya. Sonin*, the first formulas of this kind were introduced back in 1812 by Brodsky, which was then an object of study of many mathematicians, including Russians.

Not entering into any investigation of various changes in these formulas, we noted that a more expedient formula is one which was presented by Bertram in his classical "The Treatise of Differential Calculus" in 1864, also presented in the first volume of E. Hurse, "Analysis Course" in exercises named Fà de Bruneau.

This formula, giving an expression of a full derivative of any order of n from the function z by parameter t, can be expressed in the following manner:

$$\frac{d^n z}{dt^n} = n! \sum_{k-1}^{n} Z_k^{(n)}, \tag{1}$$

where

$$Z_k^{(n)} = \frac{d^k f(x)}{dx^k} \cdot X_k^{(n)}. \tag{2}$$

*N. Ya. Sonin: Concerning derivative functions of higher orders. Bulletin of Academy of Sciences No. 4, 1894. In this article is also an incomplete bibliography.

The coefficients with sequentive derivations from function f(x) with variable x, i.e. $X_k^{(n)}$ quantities, do not depend on the appearance of function f(x) but only on the sequential derivations of function x by parameter t. This is determined by the following general formula:

$$X_k^{(n)} = \sum \frac{1}{m_1! m_2! \ldots m_p!} \left(\frac{x'}{1!}\right)^{m_1} \left(\frac{x''}{2!}\right)^{m_2} \cdots \left(\frac{x^{(p)}}{p!}\right)^{m_p}, \tag{3}$$

where $p = n - k + 1,$

$x', x'', \ldots, x^{(p)}$ the essence of sequential derivations from x by t and the sum is distributed on all whole non-negative values of index $m_1, m_2, \ldots; m_p$, satisfied by

$$\left.\begin{aligned} m_1 + m_2 + m_3 + \cdots + m_{p-1} + m_p &= k \\ 1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 + \cdots (p-1)\, m_{p-1} + p \cdot m_p &= n. \end{aligned}\right\} \tag{4}$$

By these formulas, whose conclusion we do not find necessary to deduce and whose justification is easily established by a method of total induction, can be computed directly by a derivative of any order from z by parameter t. This calculation leads to a final solution of two dio-fantic equations (4), all solutions for the given m and k are easily found by a method of systematic selection.

With given function x, the quantity $X_k^{(n)}$ appears presumably also a function from t and has an operational character for the designation of formula (3) in the following manner:

$$X_k^{(n)}\{x(t)\} = \sum \prod_{s=1}^{n-k+1} \frac{1}{m_s!} \left[\frac{1}{s!} \cdot \frac{d^s x(t)}{dt^s}\right]^{m_s}$$

with conditions

$$\sum_{s=1}^{n-k+1} m_s = k, \quad \sum_{s=1}^{n-k+1} s m_s = n.$$

For application purposes, which were mentioned earlier in the article, it is more expedient to examine quantity $X_k^{(n)}$ as function p in independent variables x', x'', ..., $x^{(p)}$, then formula (3) will be written as

$$x_s = \frac{1}{s!} \cdot \frac{d^s x}{dt^s} \tag{5}$$

and examining $X_k^{(n)}$ as a function p of independent variables x_1, x_2, ..., x_p, then formula (3) will be written as

$$X_k^{(n)}\{x_1, x_2, \ldots, x_p\} = \sum \frac{x_1^{m_1} x_2^{m_2} \ldots x_p^{m_p}}{m_1!\, m_2! \ldots m_p},$$

or more simply:

$$X_k^{(n)} = \sum \frac{x_1^{m_1} x_2^{m_2} \ldots x_p^{m_p}}{m_1!\, m_2! \ldots m_p!}. \tag{6}$$

In this manner, $X_k^{(n)}$ is a whole, uniform function of k degree from p = n - k + 1 variables with reational coefficients whose reverse values are in essence, whole positive numbers. The number of members in this expression which evidently depends on n and k and equal to the various solutions of diofantic systems (4) should be designated N_k^n. With the solution of equation from (4), it is useful to know this ahead of time and we later show themethod of its determination.

We will point out easily computed instances of formula (6).

We have for every n the following formulas:

1 $X_1^{(n)} = x_n;$

2 $X_2^{(n)} = \frac{1}{2}\sum_{\sigma=1}^{n-1} x_\sigma x_{n-\sigma};$

3 $X_{n-1}^{(n)} = \frac{1}{(n-2)!} x_1^{n-2} x_2;$

4 $X_n^{(n)} = \frac{1}{n!} x_1^n.$

The second of these formulas is easier expressed in this way

$$X_2^{(n)} = \sum_{\sigma=1}^{\bar{n}} 1_n^{\sigma} x_\sigma x_{n-\sigma},$$

where $\bar{n}$ is a somewhat more while number, concluding in n/2 i.e.,

and

$$1_n^{\bar{n}} = \frac{1}{2}, \text{ if n is even,}$$

$$1_n^{\bar{n}} = 1, \text{ in all other cases.}$$

From these formulas emerge corresponding values of N_k^n in these particular cases, and specifically

$$N_1^n = 1; \quad N_2^n = E\left(\frac{n}{2}\right); \quad N_{n-1}^n = 1; \quad N_n^n = 1.$$

We will now demonstrate how to expediently compute expressions for functions $X_k^{(n)}$ for all intermediate values of k and how to find corresponding numbers of N_k^n.

As previously noted, $X_k^{(n)}$ is a whole, uniform function of the k degree in relation to the variables $x_1, x_2, \ldots, x_p$. Besides this, $X_k^{(n)}$ possesses a uniform quality in relation to the lower values of these quantities, since $1 \cdot m_1 + 2m_2 + \ldots + p \cdot m_p = n$, i.e. to the number of differential derivations. From these qualities flows a rapid, simple and recurrent correlation for function $X_k^{(n)}$. Actually, substituting to $X_k^{(n)}$, which is determined by formula (6), Eyleis theorem concerning uniform functions can be written as

$$kX_k^{(n)} = x_1 \frac{\partial X_k^{(n)}}{\partial x_1} + x_2 \frac{\partial X_k^{(n)}}{\partial x_2} + \ldots + x_p \frac{\partial X_k^{(n)}}{x_p} = \sum_{\sigma=1}^{p} x_\sigma \frac{\partial X_k^{(n)}}{\partial x_\sigma}.$$

However, it is easy to see that $\frac{\partial X_k^{(n)}}{\partial x_\sigma}$ is also a whole, uniform function of a K - 1 degree and of the same structure as the $X_k^{(n)}$. Therefore, this derivative is a function of the same type, but with a lower K - 1 value. Since a further order of differentiation of each member $X_k^{(n)}$ is the same and equal to r, then the differentiation order in the derivative will be equal to n - σ, since the first multiple is the result of σ differentiation.

In this way,
$$\frac{\partial X_k^{(n)}}{\partial x_\sigma} = X_{k-1}^{(n-\sigma)}$$

and the preceeding formula becomes
$$X_k^{(n)} = \frac{1}{k}\sum_{(\sigma=1)}^{n-k+1} x_\sigma X_{k-1}^{(n-\sigma)}, \qquad (7)$$

which gives the desired recurrent calculation and this allows calculation of the function of $X_k^{(n)}$.

For example:
$$X_2^{(n)} = \frac{1}{2}\sum_{\sigma=1}^{n-1} x_\sigma X_1^{(n-\sigma)} = \frac{1}{2}\sum_{\sigma=1}^{n-1} x_\sigma x_{n-\sigma}$$
$$X_3^{(n)} = \frac{1}{3}\sum_{\sigma=1}^{n-2} x_\sigma X_2^{(n-\sigma)}$$
$$X_4^{(n)} = \frac{1}{4}\sum_{\sigma=1}^{n-3} x_\sigma X_3^{(n-\sigma)}$$

Then, for example, applying these formulas in function $X_3^{(9)}$, we find this after making all mandatory presentations
$$X_3^{(9)} = x_1x_2x_6 + x_1x_3x_5 + x_2x_3x_4 + \frac{1}{2}x_1^2x_7 + \frac{1}{2}x_1x_4^2 + \frac{1}{2}x_2^2x_5 + \frac{1}{6}x_3^3,$$

which is easy to ckeck with direct calculations by formulas (3) and (4).

There only remains the determination of the method of finding the N_k^n numbers. This method we found in the classical expressions of Euler--"Introduction Into the Analysis of Infinitismals."* In chapter XVI, where it is shown that the desired number is equal to the coefficient with z^k x^n in the expansion of infinite products
$$\prod_{s=1}^{\infty}\frac{1}{1-zx^s} = \frac{1}{(1-zx)(1-zx^2)(1-zx^3)\dots}$$

*See Russian translation under the direction of Prof. S. Ya. Lurye, OHNTI 1936.

into an order situated by increasing degrees of z and x, in this manner we obtain

$$\prod_{s=1}^{\infty}\frac{1}{1-zx^{s}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}N_{k}^{n}z^{k}x^{n}. \tag{8}$$

From this formula it is easy to bring out a recurrent correlation for the N_k^n numbers. Actually multiplying both parts of equation (8) by 1 - 2x, we have

$$\prod_{s=1}^{\infty}\frac{1}{1-zx^{s+1}}=(1-zx)\sum_{n,k}N_{k}^{n}z^{k}x^{n}=\sum_{n,k}(N_{k}^{n}-N_{k-1}^{n-1})\,z^{k}x^{n}.$$

On the other hand, substituting z for xz in formula (8) we have

$$\prod_{s=1}^{\infty}\frac{1}{1-zx^{s+1}}=\sum_{n,k}N_{k}^{n}z^{k}x^{n+k}=\sum_{n,k}N_{k}^{n-k}z^{k}x^{n},$$

from where

$$\sum_{n,k}(N_{k}^{n}-N_{k-1}^{n-1})\,z^{k}x^{n}=\sum_{n,k}N_{k}^{n-k}z^{k}x^{n}.$$

The comparison of coefficients with identical derivatives z^k x^n in the left and right portions of the equation gives

$$N_{k}^{n}-N_{k-1}^{n-1}=N_{k}^{n-k},$$

from where

$$N_{k}^{n}=N_{k-1}^{n-1}+N_{k}^{n-k}. \tag{9}$$

This is the desired recurrent correlation which allows rapid calculation of a number of various solutions of equations (4), that is, the number of various members of uniform functions of X_k^n.

For example, for function $X_3^{(9)}$, we obtain

$$N_{3}^{9}=N_{2}^{8}+N_{3}^{6}=E\left(\frac{8}{2}\right)+N_{3}^{6}$$
$$N_{3}^{6}=N_{2}^{5}+N_{3}^{3}=E\left(\frac{5}{2}\right)+1$$

and finally

$$N_{3}^{9}=E\left(\frac{8}{2}\right)+E\left(\frac{5}{2}\right)+1=4+2+1=7.$$

With the aid of formula (9), it is easy to compile a table of

N_k^n numbers and also brings out a number of other correlations

$$N_k^n = N_{k+1}^{n+1} - N_{k+1}^{n-k},$$

$$N_k^n = \sum_{s=1}^{k} N_s^{n-k},$$

$$N_k^n = N_{n-k}^{2n-2k} - \sum_{s=0}^{n-2k-1} N_{n-k-1}^{n-k}.$$

We note that with $n - k < k$ or $k > n/2$ we obtain

$$N_k^{n-k} = 0,$$

and then

$$N_k^n = N_{k-1}^{n-1},$$

which is easily checked immediately.

Of further interest is the discovery of the totals of N_k^n for a given n, that is, the number of all members in formula (1).

To obtain this number, which depends only on n, let us arrange series (8) by x degrees. We receive

$$\prod_{s=1}^{\infty} \frac{1}{1 - zx^s} = \sum_{n=0}^{\infty} A_n(z) \cdot x^n, \tag{10}$$

where

$$A_n(z) = \sum_{k=1}^{n} N_k^n z^k,$$

from which, assuming that z = 1, we obtain

$$\sum_{k=1}^{n} N_k^n = A_n(1) = A_n. \tag{11}$$

But, assuming in formula (10) that z = 1, we obtain

$$\prod_{s=1}^{\infty} \frac{1}{1 - x^s} = \sum_{n=0}^{\infty} A_n x^n, \tag{12}$$

it follows that the desired number A_n equal to the coefficient with x^n in an expansion of infinite derivations

$$\frac{I}{(1 - x)(1 - x^2)(1 - x^3)\ldots}$$

in a series distributed by increasing degrees of x.

For the direct determination of A_n numbers, let us examine the following expansion

$$\prod_{s=1}^{\infty}(1-x^s)=\sum_{n=0}^{\infty}a_n x^n, \tag{13}$$

Multiplying equations (12) and (13) we obtain

$$\sum_{n=0}^{\infty}(A_0a_n+A_1a_{n-1}+\ldots+A_{n-1}a_1+A_na_0)\,x^n=1,$$

from where

$$A_0a_0=1$$
$$A_0a_n+A_1a_{n-1}+\ldots+A_{n-1}a_1+A_na_0=0,$$

which gives the recurrent formula for the calculation of A_n

$$A_n=-a_n-\sum_{s=1}^{n-1}A_s a_{n-s}. \tag{14}$$

Concerning a_n numbers, in the previously mentioned Euler's expression it shows that $a_n=(-1)^x$,

when n is a number appearing is $n=\frac{1}{2}(3x^2\mp x)\quad(x=0,\ 1,\ 2\ldots)$

and $a_n = 0$

in all remaining cases.

In this way, all coefficients of a_n are known and formula (14) consequently allows finding the A_n numbers independent from N_k^n numbers, which may serve as a control for the calculation.

It is easy to obtain for A_n, expressions clearly dependent only on a_n numbers. Actually we can write the following systems of equations

$$\begin{aligned}
a_0A_n+a_1A_{n-1}+a_2A_{n-2}+a_3A_{n-3}+\ldots+a_{n-3}A_3+a_{n-2}A_2+a_{n-1}A_1+a_nA_0&=0\\
a_0A_{n-1}+a_1A_{n-2}+a_2A_{n-3}+\ldots+a_{n-4}A_3+a_{n-3}A_2+a_{n-2}A_1+a_{n-1}A_0&=0\\
a_0A_{n-2}+a_1A_{n-3}+\ldots+a_{n-5}A_3+a_{n-4}A_2+a_{n-3}A_1+a_{n-2}A_0&=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
a_0A_3+a_1A_2+a_2A_1+a_3A_0&=0\\
a_0A_2+a_1A_1+a_2A_0&=0\\
a_0A_1+a_1A_0&=0\\
a_0A_0&=1
\end{aligned}$$

Eliminating from these equations A_0, $A_1 \ldots$, A_{n-2}, A_{n-1} and keeping in mind that the system determination is equal to $a_0^{n+1} = 1$, we obtain:

$$A_n = \begin{vmatrix} 0 & a_1 & a_2 & a_3 \ldots\ldots & a_{n-3} & a_{n-2} & a_{n-1} & a_n \\ 0 & a_0 & a_1 & a_2 \ldots\ldots & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \\ 0 & 0 & a_0 & a_1 \ldots\ldots & a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} \\ 0 & 0 & 0 & a_0 \ldots\ldots & a_{n-6} & a_{n-5} & a_{n-4} & a_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0\;0 & 0 & 0 & 0 \ldots\ldots & a_0 & a_1 & a_2 & a_3 \\ 0\;0 & 0 & 0 & 0 \ldots\ldots & 0 & a_0 & a_1 & a_2 \\ 0\;0 & 0 & 0 & 0 \ldots\ldots & 0 & 0 & a_0 & a_1 \\ 1\;0 & 0 & 0 & 0 \ldots\ldots & 0 & 0 & 0 & a_0 \end{vmatrix}$$

Since A_n numbers depend only on one index, ~~then~~ it is simpler to compute a table for them, than for N_k^n numbers which depend on two indeces. This table is obtained if we directly write out expansion (12):

$$\prod_{s=1}^{\infty} \frac{1}{1-x^s} = \sum_{n=0}^{\infty} A_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \\ + 22x^8 + 30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} + 176x^{15} + \\ + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} + 792x^{21} + 1002x^{22} + \\ + 1255x^{23} + 1575x^{24} + 1958x^{25} + 2436x^{26} + 3010x^{27} + 3718x^{28} + \\ + 4565x^{29} + 5604x^{30} + \ldots \qquad (15)$$

Utilizing formula (14), it is easy to continue this series to a sufficient length. Let us turn our attention to the N_k^n number which may be expressed through the coefficients of expansion (15) A_n.

Let us notice, first of all, that from the very determination of A_n numbers it follows that any A_ν is a number of whole non-negative solutions of Disphantine equations

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + \nu\alpha_\nu = \nu.$$

Let us return to equation (4) from which we obtain

$$m_2 + 2m_3 + 3m_4 + \ldots + (n-k)\, m_{n-k+1} = n - k.$$

Evidently the ~~A_n~~ A_{N-K} number represents a number of whole, ~~uniform~~, non-negative ~~number of~~ solutions to this equation and ~~each of these solutions~~ the unique value m_1 each of solutions corresponds to formulas (4) a single value of m_1

$$m_1 = k - (m_2 + m_3 + \ldots + m_{n-k+1}).$$

If the m_1 value calculated by this formula is not negative, then we can obtain a certain system of solving ~~these~~ equations (4). But $m_2 + m_3 + \ldots + m_{n-k+1} < n - k$, and therefore, m_1 will be predictably non-negative if $n - k \leq k$, that is if $k \geq n/2$.

Therefore we obtain

$$N_k^n = A_{n-k}, \qquad \text{if } 2k \geqslant n.$$

If $2k < n$, then the last formula is not ~~usable~~ applicable, but in this case other formulas may be obtained. Actually, we will rewrite correlation (9) as

$$N_k^n = N_{k+1}^{n+1} - N_{k+1}^{n-k},$$

from which we obtain

$$N_k^n = N_{n-k}^{2n-2k} - \sum_{\sigma=0}^{n-2k-1} N_{n-k-\sigma}^{n-k},$$

then we have

$$N_k^n = A_{n-k} - A_0 - A_1 - A_2 - \ldots$$

But this formula ends in different ways, depending on the value of k (which is assumed to be less than $n/2$). Without describing the details, we present only the final formulas. We have

$$N_k^n = A_{n-k}, \qquad \text{if } k \geq n/2$$

$$N_k^n = A_{n-k} - \sum_{\sigma=0}^{n-2k-1} A_\sigma \qquad \text{if } \frac{n-2}{3} \leqslant k < \frac{n}{2},$$

while $\bar{n} = E\left(\frac{n-k}{2}\right)$; $N_k^n = A_{n-k} - \sum_{\sigma=0}^{\bar{n}} A_\sigma - \sum_{\sigma=\bar{n}+1}^{n-2k-1} N_{n-k-\sigma}^{n-k}$, if $k < \frac{n-2}{3}$,

These formulas are sufficient for a rapid calculation of A_k^n numbers with the aid of coefficients the order of (15).

For example:

$$N_5^{17} = A_{12} - \sum_{\sigma=0}^{6} A_\sigma = 77 - 1 - 1 - 2 - 3 - 5 - 7 - 11 = 47$$

$$N_6^{23} = A_{17} - \sum_{\sigma=0}^{8} A_\sigma - \sum_{\sigma=9}^{10} N_{17-\sigma}^{17} = A_{17} - A_0 - A_1 - A_2 - A_4 - A_5 - A_6 -$$
$$- A_7 - A_8 - N_8^{17} - N_7^{17} = A_{17} - A_0 - A_1 - A_2 - A_3 - A_4 - A_5 - A_6 - A_7 -$$
$$- A_8 - (A_9 - A_0) - (A_{10} - A_0 - A_1 - A_2) = A_{17} + A_0 - A_3 - A_4 - A_5 -$$
$$- A_6 - A_7 - A_8 - A_9 - A_{10} = 297 + 1 - 3 - 5 - 7 - 11 - 15 - 22 - 30 -$$
$$- 42 = 163.$$

With this we finish the examination of the simplest instance where the complex function depends only on one intermediate variable.

II

Now let us examine function~~s~~ of two independent variables.

~~differentiating~~ z = f(x, y),

differentiable often ~~any number~~ by each of the variables x and y which are assumed to be differentiated the required ~~any~~ number of times by the functions of parameter t.

Then z will be a complex function of parameter t and its complete derivation by t of any order n may be calculated by the following formula, which is a true generalization of formula (1) and whose justification is easily found by a method of complete induction:

$$\frac{d^n z}{dt^n} = n! \sum_{k=1}^{n} Z_k^{(n)}, \tag{16}$$

where

$$Z_k^{(n)} = \sum_{s=0}^{k} \frac{\partial^k f(x, y)}{\partial x^{k-s} \partial y^s} \cdot Z_{k-s,\, s}^{(n)} \tag{17}$$

Coefficients $Z_{k-s,s}^{(n)}$ with partial derivati~~ons~~ves do not depend on the form of function~~s~~ f(x, y) but only on sequential derivati~~ons~~ves from x and y by parameter t. Utilizing the designation ~~of~~ (5), we can present these quantities in the following general formula

$$Z^{(n)}_{k-s,\,s} = \sum \frac{x_1^{m_1} \cdot x_2^{m_2} \cdot \ldots \cdot x_p^{m_p}\, y_1^{n_1}\, y_2^{n_2} \cdot \ldots \cdot y_p^{n_p}}{m_1!\, m_2! \cdots m_p!\, n_1!\, n_2! \cdots n_p!} \tag{18}$$

where $p = n - k + 1$,

and the total is distributed in all non-negative values of indeces

$m_1, m_2, \ldots, m_p, n_1, n_2 \ldots n_p,$ satisfying the conditions:

$$\left.\begin{array}{c} m_1 + m_2 + \ldots + m_p = k - s,\; n_1 + n_2 + \ldots + n_p = s \\ m_1 + 2m_2 + 3m_3 + \ldots pm_p + n_1 + 2n_2 + 3n_3 + \ldots + pn_p = n \end{array}\right\} \tag{19}$$

By these formulas it is easy to directly calculate these functions of $Z^{(n)}_{k-s,s}$. Thus, we have first of all

$$Z^{(n)}_{k,0} = X^{(n)}_k \quad Z^{(n)}_{0,k} = Y^{(n)}_k, \tag{20}$$

where, evidently the y functions are ~~value is~~ calculated by these same formulas, as are the x functions, with a change of all x's to y's.

It is also easy to directly obtain the ~~this~~ formula

$$Z^{(n)}_{1,1} = \sum_{\sigma=1}^{n-1} x_{n-\sigma} y_\sigma . \tag{21}$$

For other values of k and s we have the following easily verifiable ~~checked~~ formulas

$$Z^{(n)}_{k-s,\,s} = \sum_{\sigma=s}^{n-k+s} X^{(n-\sigma)}_{k-s} \cdot Y^{\sigma}_{s} \tag{22}$$

or

$$Z^{(n)}_{k-s,\,s} = \sum_{\sigma=k-s}^{n-s} X^{(\sigma)}_{k-s} \cdot Y^{(n-\sigma)}_{s}. \tag{23}$$

From these formulas it is easy to calculate the number of various members contained in function $Z^{(n)}_{k-s,s}$. Actually, if we designate this number by ~~through~~ $N^n_{k-s,s}$, then we will evidently have

$$N^n_{k-s,\,s} = \sum_{\sigma=s}^{n-k+s} N^{n-\sigma}_{k-s} \cdot N^{\sigma}_{s} = \sum_{\sigma=k-s}^{n-s} N^{\sigma}_{k-s} N^{n-\sigma}_{s}. \tag{24}$$

and ~~while~~ the number of various members, entering into the expression of function Z^n_k would be evidently equal to

$$\overline{N}^n_k = \sum_{s=0}^{k} N^n_{k-s,\,s} = 2\sum_{s=0}^{\bar{k}} 1^s_k N^n_{k-s,\,s}, \tag{25}$$

where $\bar{k}=E\left(\frac{k}{2}\right)$.

Formulas (22) and (23) are quite useful for the calculation of function z and, in this way, for the very ~~it is a~~ rapid compilation of ~~derivative~~ an expression of a derivative of any order with a complex function with two intermediate arguments. Actually, the calculation of function z requires only the ability to calculate the x functions, which was examined in detail in section I.

Paragraph of Russian is omitted here.

These recurrent formulas follow from the ~~a~~ general formula (18) which shows that function $Z^{(n)}_{k-s,s}$ is a uniform function of ~~a~~ k degree ~~s~~ from 2p ~~of~~ variables $x_1 \ldots x_p$, $y_1 \ldots y_p$.

Therefore, again utilizing Euler's theory concerning uniform functions as was done in section I, we easily obtain a recurrent formula of the type

$$Z^{(n)}_{k-s,s}=\frac{1}{k}\sum_{\sigma=1}^{p}\{x_\sigma Z^{(n-\sigma)}_{k-1-s,s}+y_\sigma Z^{(n-\sigma)}_{k-s,s-1}\}, \tag{26}$$

and utilizing this formula ~~a~~ a sufficient number of times, we will derive ~~present~~ the ~~cal-~~culations of the necessary function z with any two lower values to the computation of function z. ~~Since~~ in which either one of two lower values is equal to zero each of the two lower values is equal to one, then we directly utilize formulas (20) and (21).

Finally, function z can be examined as a uniform function of k - s degree from variables x_1, $x_2 \ldots x_p$, or as ~~since~~ the uniform function of s degree of ~~is~~ from variables y_1, $y_2 \ldots y_p$.

Utilizing Euler's theorem in this way, again as in section I, we easily obtain two more recurrent formulas, quite useful for various instances of computing function z:

$$Z^{(n)}_{k-s,s}=\frac{1}{k-s}\sum_{\sigma=1}^{p}x_\sigma Z^{(n-\sigma)}_{k-1-s,s} \tag{27}$$

$$Z^{(n)}_{k-s,\,s} = \frac{1}{s}\sum_{\sigma=1}^{p} y_\sigma\, Z^{(n-\sigma)}_{k-s,\,s-1}\,. \tag{28}$$

To give examples of utilizing the presented recurrent formulas, we will examine certain very simple cases, which are encountered by us in other works.

Let z = x · y,

where x and y are the essence of any functions of parameter t. Then:

$$\frac{\partial z}{dx} = y, \quad \frac{\partial z}{dy} = x, \quad \frac{\partial^2 z}{\partial x^2} = 0, \quad \frac{\partial^2 z}{\partial x \partial y} = 1, \quad \frac{\partial^2 z}{\partial y^2} = 0$$

and all remaining partial derivatives are in essence nil.

Formula (16) gives

$$\frac{d^n z}{dt^n} = n!\,(Z_1^{(n)} + Z_2^{(n)}).$$

From formula (17) we have

$$Z_1^{(n)} = yZ^{(n)}_{1,0} + xZ^{(n)}_{0,1} = yX_1^{(n)} + xY_1^{(n)} = yx_n + xy_n.$$

$$Z_2^{(n)} = Z^{(n)}_{1,1} = \sum_{\sigma=1}^{n-1} x_{n-\sigma}\, y_\sigma\,.$$

Assuming that for symmetry $x = x_0$, $y = y_0$,

we obtain

$$\frac{d^n(xy)}{dt^n} = n!\sum_{\sigma=0}^{n} x_{n-\sigma}\, y_\sigma\,. \tag{29}$$

If we substitute x_{n-} and y for these expressions, then we will obtain a Lubnitz formula for derivatives of the nth order from the derivative of two functions.

Let $z = xy^2$,

where x and y are in essence any functions of parameter t. Then:

$$\frac{\partial z}{\partial x} = y^2, \quad \frac{\partial z}{dy} = 2xy, \quad \frac{\partial^2 z}{\partial x^2} = 0, \quad \frac{\partial^2 z}{\partial x \partial y} = 2y, \quad \frac{\partial^2 z}{\partial y^2} = 2x$$

$$\frac{\partial^3 z}{\partial x^3} = 0, \quad \frac{\partial^3 z}{\partial x^2 \partial y} = 0, \quad \frac{\partial^3 z}{\partial x \partial y^2} = 2, \quad \frac{\partial^3 z}{\partial y^3} = 0$$

and all remaining partial derivatives are in essence, nil.

Formula (16) gives: $\frac{d^n z}{dt^n} = n!(Z_1^{(n)} + Z_2^{(n)} + Z_3^{(n)})$.

From formula (17) we obtain:

$$Z_1^{(n)} = y^2 Z_{1,0}^{(n)} + 2xyZ_{0,1}^{(n)} = y^2 x_n + 2xy\,y_n,$$

$$Z_2^{(n)} = 2y\,Z_{1,1}^{(n)} + 2x\,Z_{0,2}^{(n)} = 2y\sum_{\sigma=1}^{n-1} x_{n-\sigma}\,y_\sigma + 2x\sum_{\sigma=1}^{\bar{n}} 1_n^\sigma y_\sigma\,y_{n-\sigma}, \qquad Z_3^{(n)} = 2\,Z_{1,2}^{(n)}.$$

For the calculation of the last function, it is easier to implement formula (27) which gives

$$Z_{1,2}^{(n)} = \sum_{\sigma=1}^{n-2} x_\sigma Z_{0,2}^{(n-\sigma)} = \sum_{\sigma=1}^{n-2} x_\sigma U_2^{(n-\sigma)} = \frac{1}{2}\sum_{\sigma=1}^{n-2} x_\sigma \sum_{\tau=1}^{n-\sigma-1} y_\tau y_{n-\sigma-\tau}.$$

Making the summation and again assuming $x = x_0$, $y = y_0$, we easily obtain the following formula:

$$\frac{d^n (xy^2)}{dt^n} = 2n!\sum_{k=0}^{n} x_{n-k}\sum_{\sigma=0}^{\bar{k}} 1_k^\sigma\,y_{k-\sigma}\,y_\sigma, \tag{30}$$

where $\bar{k}$ signifies a larger whole number contained in k/2, that is $\bar{k} = E(k/2)$.

Without demonstrating any further examples, we will merely point out that a curious formula for n derivative fractions, which is easy to obtain when examining this function

$$Z = y/x.$$

This formula, which we will not explain in detail, may be written in the following manner:

$$\frac{d^n}{dt^n}\left(\frac{y}{x}\right) = n!\sum_{k=1}^{n}\frac{(-1)^{k+1}(k-1)!}{x^{k+1}}\sum_{\sigma=1}^{n-k+1}(xy_\sigma - yx_\sigma)\,X_{k-1}^{(n-\sigma)}.$$

Let us examine one more important instance of a complex function, when it depends on the independent variable t and is directly in consequence of one intermediate variable.

We will have this instance assuming simply that y = t. Then

$$Z = f(x,t)$$

where x is any function from t.

In this case $y_1 = 1,\ y_2 = y_3 = \dots = 0.$

Formulas (16) and (17) give

$$\frac{d^n f(x,t)}{dt^n} = n! \sum_{k=1}^{n} \sum_{s=0}^{k} \frac{\partial^k f(x,t)}{\partial x^{k-s} \partial t^s} Z^{(n)}_{k-s,s}.$$

but $Z^{(n)}_{k,0} = X^{(n)}_k$,

while for other values of s, we will utilize (28) which gives

$$Z^{(n)}_{k-s,s} = \frac{1}{s} Z^{(n-1)}_{k-s,s-1},$$

again utilizing formula (28) and the s order, we will have

$$Z^{(n)}_{k-s,s} = \frac{1}{s!} Z^{(n-s)}_{k-s,0} = \frac{1}{s!} X^{(n-s)}_{k-s},$$

as a result the desired formula may be written in the following final manner:

$$\frac{d^n f(x,t)}{dt^n} = n! \sum_{k=1}^{n} \sum_{s=0}^{k} \frac{\partial^k f(x,t)}{\partial x^{k-s} \partial t^s} \cdot \frac{1}{s!} X^{(n-s)}_{k-s}. \tag{31}$$

The last formula may be useful with the solution of the following problem, often encountered in the addition.

Let us equate f(x,t) = 0,

having t = 0 with a solution x = 0. If $f'x(0,0) \neq 0$, then with $t \neq 0$ sufficiently small, the solution of our equation may be presented in the order $x = x_1 t + x_2 t^2 + \dots + x_n t^n + \dots$

The coefficients of this order are, in essence,

$$x_n = \frac{1}{n!} \left(\frac{d^n x}{dt^n} \right)_{t=0}.$$

With the aid of formula (31), it is easy to obtain a recurrent formula for the solution of these coefficients.

Actually, differentiating the equation f(x,t) = 0 n times by t, considering x as a function of t and assuming after differentiation t = 0, x = 0, we obtain as the basis of (31)

$$\sum_{k=1}^{n} \sum_{s=0}^{k} f^{(k)}_{k-s,s}(0,0) \cdot \frac{1}{s!} X^{(n-s)}_{k-s} = 0,$$

where X is the essence of functions from x_1, x_2,... From here we may express

$$f'_x(0,0)x_n + \sum_{k=2}^{n}\sum_{s=0}^{k} f^{(k)}_{k-s,s}(0,0)\cdot\frac{1}{s!}X^{(n-s)}_{k-s} = 0,$$

from which

$$x_n = -\frac{1}{f'_x(0,0)}\sum_{k=2}^{n}\sum_{s=0}^{k} f^{(k)}_{k-s,s}(0,0)\cdot\frac{1}{s!}X^{(n-s)}_{k-s}, \tag{32}$$

this is expressed by coefficient x_n with t^n through all preceding x_1, x_2,..., x_{n-1}.

In this way, formula (32) allows calculation of the coefficients of the necessary expansion.

III

The formulas examined in the previous section can be easily distributed in an instance of any number of intermediate variables.

Let us have the following function: $Z=f(x_1, x_2, \ldots, x_r)$

from r the independent variables x_1, x_2,..., x, is the essence of function of parameter t, differentiated any number of times, then z will be the complex function from t and its complete variable of the nth order can be calculated by the following formula:

$$\frac{d^n z}{dt^n} = n!\sum_{k=1}^{n} Z^{(n)}_k, \tag{33}$$

where

$$Z^{(n)}_k = \sum \frac{\partial^k f(x_1, x_2, \ldots, x_r)}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_r^{k_r}}\cdot Z^{(n)}_{k_1, k_2, \ldots k_r}; \tag{34}$$

The total is distributed in all whole, non-negative values of the indeces k_1, k_2,...,k_r, which satisfy the conditions

$$k_1 + k_2 + \ldots + k_r = k,$$

and therefore contains as many members as can be found in partial derivatives of a k order from function r with independent variables. The latter, as it is known, contains as many members, as is contained in a complete, uniform multinomial of a k degree with an unknown r which is

$$\frac{(k+1)(k+2)(k+3)\ldots(k+r-1)}{1.2.3\ldots(r-1)}$$

members. That many members are contained in formula (34), determining function $Z_k^{(n)}$.

The coefficients with partial derivatives in formula (34), that is, functions $Z_{k1}^{(n)}$, $k_2, \ldots, k_r$, as in preceeding more simple cases, do not depend on functions $f(x_1, x_2, \ldots, x_r)$, but only on sequential derivatives functions $x_1, x_2, \ldots x_r$, by parameter t until a certain number determined by an order of n and k.

These coefficients can be calculated by the following general formula, which is a true generalization of formula (18) of the previous section:

$$Z^{(n)}_{k_1, k_2, \ldots, k_r} = \sum \frac{\left(\frac{x_1'}{1!}\right)^{m_1^{(1)}} \left(\frac{x_1''}{2!}\right)^{m_2^{(1)}} \cdots \left(\frac{x_1^{(p)}}{p!}\right)^{m_p^{(1)}} \left(\frac{x_2'}{1!}\right)^{m_1^{(2)}} \left(\frac{x_2''}{2!}\right)^{m_2^{(2)}} \cdots}{m_1^{(1)}!\, m_2^{(1)}! \ldots m_p^{(1)}!\, m_1^{(2)}!\, m_2^{(2)}! \ldots} \to$$

$$\leftarrow \frac{\cdots \left(\frac{x_2^{(p)}}{p!}\right)^{m_p^{(2)}} \cdots \left(\frac{x_r^1}{1!}\right)^{m_1^{(r)}} \left(\frac{x_r''}{2!}\right)^{m_2^{(r)}} \cdots \left(\frac{x_r^{(p)}}{p!}\right)^{m_p^{(r)}}}{\ldots m_p^{(2)}!\, m_1^{(r)}!\, m_2^{(r)}! \ldots m_p^{(r)}!},$$

where as usual,

$$p = n - k + 1$$

and the total is distributed in all whole, non-negative index values.

$$m_1^{(1)},\ m_2^{(1)},\ \ldots,\ m_p^{(1)},\ m_1^{(2)},\ m_2^{(2)}, \ldots, m_p^{(2)}, \ldots, m_1^{(r)},\ m_2^{(r)}, \ldots, m_p^{(r)},$$

satisfying conditions:

$$\begin{array}{ll}
m_1^{(1)} + m_2^{(1)} + \ldots + m_p^{(1)} = k_1; & m_1^{(1)} + 2m_2^{(1)} + \ldots + pm_p^{(1)} + \\
m_1^{(2)} + m_2^{(2)} + \ldots + m_p^{(2)} = k_2; & + m_1^{(2)} + 2m_2^{(2)} + \ldots + pm_p^{(2)} + \\
\ldots\ldots\ldots\ldots\ldots & + \ldots\ldots\ldots\ldots\ldots + \\
m_1^{(r)} + m_2^{(r)} + \ldots + m_p^{(r)} = k_r; & + m_1^{(r)} + 2m_2^{(r)} + \ldots + pm_p^{(r)} = n
\end{array}$$

In this manner, each of the functions $Z^{(n)}_{k_1, k_2, \ldots, k_r}$ $(k_1 + k_2 + \cdots + K_r = K$ may be calculated independently from all others, with the aid of a uniform and elementary process of solving systems r + 1 diophantic equations with r · p as unknown.

These equations are easily solved with the aid of systematic selection, however with a large number of unknowns, this selection demands much time and effort and is therefore not practically suitable for this operation.

Another method of calculating functions $Z^{(n)}_{k_1, k_2, \ldots, k_r}$ $(k_1 + k_2 + \ldots + k_r = K$ is based on the application multi-numbered recurrent correlations connecting the necessary function z with functions of the same type. But these are lesser values of the upper index n or lesser values of lower indeces k_1, $k_2, \ldots k_r$ or a smaller number of indeces. These recurrent conditions are obtained in the same conditions as corresponding formulas of preceeding sections and we limit ourselves in that we will express certain ones useful to us in some presentations.

Let us introduce, first of all, abbreviated designations similar to (5) of the first section, and specifically examining any derivative of any function x_1, $x_2, \ldots x_r$ by parameter t, we assume

$$x^{(s)}_{\sigma} = \frac{1}{\sigma!} \cdot \frac{d^{\sigma} x_s}{dt^{\sigma}}. \tag{35}$$

Then the expression for function $Z^{(n)}_{k_1, k_2, \ldots k_r}$ will be presented in the following manner

$$Z^{(n)}_{k_1, k_2, \ldots k_r} = \sum \frac{\left(x^{(1)}_1\right)^{m^{(1)}_1} \left(x^{(1)}_2\right)^{m^{(1)}_2} \ldots \left(x^{(1)}_p\right)^{m^{(1)}_p} \left(x^{(2)}_1\right)^{m^{(2)}_1} \ldots \left(x^{(2)}_p\right)^{m^{(2)}_p} \ldots}{m^{(1)}_1!\, m^{(1)}_2! \ldots m^{(1)}_p!\, m^{(r)}_1!\, m^{(2)}_2! \ldots m^{(2)}_p!} \cdot \frac{\ldots \left(x^{(r)}_1\right)^{m^{(r)}_1} \ldots \left(x^{(r)}_p\right)^{m^{(r)}_p}}{\ldots m^{(r)}_1! \ldots m^{(r)}_p!} \tag{36}$$

or even more abbreviated

$$Z^{(n)}_{k_1, k_2, \ldots k_r} = \sum \prod_{\rho=1}^{r} \frac{\left(x_1^{(\rho)}\right)^{m_1^{(\rho)}} \left(x_2^{(\rho)}\right)^{m_2^{(\rho)}} \cdots \left(x_p^{(\rho)}\right)^{m_p^{(\rho)}}}{m_1^{(\rho)}!\, m_2^{(\rho)}! \ldots m_p^{(\rho)}!}.$$

In this manner, function $Z^{(n)}_{k_1,\ k_2, \ldots k_r}$, is a whole, uniform function of a $k = k_1 + k_2 + \ldots + k_r$ degree from r · p of variables $x_1^{(1)}, x_2^{(1)}, \ldots, x_p^{(1)}, \ldots, x_1^{(2)}, x_2^{(2)}, \ldots x_p^{(2)}; \ldots; x_1^{(r)}, x_2^{(r)}, \ldots, x_p^{(r)}$ with rational coefficients. Function $Z^{(n)}_{k_1,\ k_2, \ldots k_r}$ may be examined as a whole, uniform function of k_σ degree from a p variable of either r group

$$x_1^{(\sigma)}, \quad x_2^{(\sigma)}, \quad \ldots x_p^{(\sigma)}, \quad (\sigma = 1, 2 \ldots, r)$$

or a while, uniform function of r · p variables, entering into any $\overline{r}$ from ($\overline{r}$ = 2, 3,...,r).

From here, with the aid of Euler's theorem concerning uniform functions one may actually obtain multi-numbered recurrent correlations similar to the one made in the conclusion of the basic recurrent formula for function x in the first section. For example

$$Z^{(n)}_{k_1, k_2, \ldots, k_r} = \frac{1}{k_\rho} \sum_{s=1}^{\rho} x_s^{(\rho)} Z^{(n-s)}_{k_1, k_2, \ldots k_\rho - 1, \ldots, k_r}, \tag{37}$$

$$Z^{(n)}_{k_1, k_2, \ldots, k_r} = \frac{1}{k} \sum_{s=1}^{p} \sum_{\rho=1}^{r} x_s^{(\rho)} Z^{(n-s)}_{k_1, k_2, \ldots, k_\rho - 1, \ldots, k_r}. \tag{38}$$

The most notable fact, however, is that function z with r indeces may be expressed through r function x of the first section, every one of which depends on p variables, entering into one and the same group.

This becomes obvious if we note that the z, r - 1 indeces whose essence is zero is a simple corresponding function of x. Actually if

$$k_1 = k, k_2 = k_3 = \ldots = k_r = 0,$$

then the formula (36) becomes

$$Z^{(n)}_{k,0,\ldots 0} = \sum \frac{(x_1^{(1)})^{m_1^{(1)}} (x_2^{(1)})^{m_2^{(1)}} \ldots (x_p^{(1)})^{m_p^{(1)}}}{m_1^{(1)}!\, m_2^{(1)}! \ldots m_p^{(1)}!},$$

where

$$m_1^{(1)} + m_2^{(1)} + \ldots + m_p^{(1)} = k$$
$$m_1^{(1)} + 2m_2^{(1)} + \ldots + pm_p^{(1)} = n.$$

In this manner, $Z^{(n)}_{k,0\ldots,0}$ is a function of x from p variables of the first group $x_1^{(1)}, x_2^{(1)}, \ldots, x_p^{(1)}$ and we can express $Z^{(n)}_{k,\,0,\,0,\ldots,0} = X_k^{(n)}\{x_1^{(1)},\ x_2^{(1)}, \ldots, x_p^{(1)}\}$

The same as if $k_1 = k_2 = \ldots k_{\rho-1} = k_{\rho+1} = \ldots k_r = 0,$
$k_\rho = k\,(\rho = 2, 3, \ldots, r),$

now

$$Z^{(n)}_{0,0,\ldots k,\ldots,0} = X_k^{(n)}\{x_1^{(\rho)},\ x_2^{(\rho)}, \ldots, x_p^{(\rho)}\}. \tag{39}$$

For convenience sake, we will not write out the variables in the brackets, but will express it this way: $Z^{(n)}_{0,0,\ldots k,\ldots,0} = \overset{\rho}{X}{}_k^{(n)}$.

If we now examine formula (26), then it will become absolutely clear that in its first part is contained the total product r of function x, each one of the variables of one group. It is not hard to see that the lower indeces of these functions are numbers k_1, k_2, ..., k_r. If we designate the upper indeces through n_1, $n_2 \ldots n_3$, then we can express a simple basic formula

$$Z^{(n)}_{k_1, k_2, \ldots, k_r} = \sum \overset{1}{X}{}_{k_1}^{(n_1)} \overset{2}{X}{}_{k_2}^{(n_2)} \ldots \overset{r}{X}{}_{k_r}^{(n_r)}, \tag{40}$$

where $k_1 + k_2 + \ldots + k_r = k$ and evidently $n_1 + n_2 + \ldots + n_r = n$. The total is distributed in all whole, positive index values of n_1, $n_2, \ldots n_r$, satisfying the evident conditions $n_1 \geqslant k_1,\ n_2 \geqslant k_2, \ldots, n_r \geqslant k_r$.

In this manner, the calculation of any function of z is presented simply to the combination of function x, the calculation methods of which are established in the first section.

It is now clear that from formula (40), one can produce many more, combining various methods of $\bar{r} < r$ from the function of x. This will give the functions of x with numbers of lower values less than r. We note only one formula of this type

$$Z^{(n)}_{k_1, k_2, \ldots, k_{r-1}, k_r} = \sum_{\sigma = k_r}^{n-k+k_r} Z^{(n-\sigma)}_{k_1, k_2, \ldots, k_{r-1}} \cdot X^{(\sigma)}_{k_r}. \qquad (41)$$

Now, if

$$x_r = t$$

then from formula (41) follows

$$Z^{(n)}_{k_1, k_2, \ldots, k_{r-1}, k_r} = \frac{1}{k_r!} Z^{(n-k_r)}_{k_1, k_2, \ldots, k_{r-1}}. \qquad (42)$$

We will not examine examples of formulas implemented by us at this time. Let us note only that with its aid, a general formula of Lubnitz may be obtained for an n derivative product of function r. This well known formula in the designation used by us, has the following expression:

$$\frac{d^n (x_1 x_2 \ldots x_r)}{dt^n} = n! \sum x_1^{(n_1)} x_2^{(n_2)} \ldots x_r^{(n_r)}, \qquad (43)$$

where the total is distributed in all whole, non-negative values of indeces n_1, n_2, ..., n_r, satisfying the condition

$$n_1 + n_2 + \ldots + n_r = n.$$

When utilizing this formula, it should be remembered that the upper values indicate the derivative order which should be divided by corresponding factors and that $X^{(0)}_\rho = X_\rho$ In particular, from formula (43) we will obtain the following

$$\frac{d^n}{dt^n} (x^r) = n! \sum x^{(n_1)} x^{(n_2)} \ldots x^{(n_r)}$$

with the same method of summation.

In conclusion, we present tables of certain x and z functions compiled by S. P. Tokmalaeva.

The following table is a basis of function x.

$$
\begin{aligned}
n=1.\quad & X_1^{(1)} = x_1, \\
n=2.\quad & X_1^{(2)} = x_2, \\
& X_2^{(2)} = \frac{1}{2} x_1^2, \\
n=3.\quad & X_1^{(3)} = x_3, \\
& X_2^{(3)} = x_1 x_2, \\
& X_3^{(3)} = \frac{1}{6} x_1^3, \\
n=4.\quad & X_1^{(4)} = x_4, \\
& X_2^{(4)} = x_1 x_3 + \frac{1}{2} x_2^2, \\
& X_3^{(4)} = \frac{1}{2} x_1^2 x_2, \\
& X_4^{(4)} = \frac{1}{24} x_1^4. \\
n=5.\quad & X_1^{(5)} = x_5, \\
& X_2^{(5)} = x_1 x_4 + x_2 x_3, \\
& X_3^{(5)} = \frac{1}{2} x_1^2 x_3 + \frac{1}{2} x_1 x_2^2, \\
& X_4^{(5)} = \frac{1}{6} x_1^3 \cdot x_2, \\
& X_5^{(5)} = \frac{1}{120} x_1^5. \\
n=6.\quad & X_1^{(6)} = x_6, \\
& X_2^{(6)} = x_1 x_5 + x_2 x_4 + \frac{1}{2} x_3^2, \\
& X_3^{(6)} = \frac{1}{2} x_1^2 x_4 + x_1 x_2 x_3 + \frac{1}{6} x_2^3, \\
& X_4^{(6)} = \frac{1}{6} x_1^3 x_3 + \frac{1}{4} x_1^2 x_2^2, \\
& X_5^{(6)} = \frac{1}{24} x_1^4 x_2, \\
& X_6^{(6)} = \frac{1}{720} x_1^6.
\end{aligned}
$$

We will limit ourselves to these expressed formulas. More detailed tables of this type are given in an article by P. T. Reznikovsky.[1]

The compilation of tables for functions of z in instances of two or larger numbers of intermediate variables is presented, as previously explained, to algebraic combinations of function x with the proper substitutions where it is necessary, changing letter x for y, z.

These calculations are simplified significantly, since the factual computations are carried out only for certain z functions, while the rest are expressed without any calculations. Let us clarify this in the case of two or three intermediate variables and present the results of the calculations made by S. P. Tokmalaeva.

For the compilation of complete derivative functions z = f(x,y) it is necessary to express the function of z with the aid of formula (22) and (23), where the function of x are taken directly from the table, while y functions are obtained from x functions in the same table, substituting the letter x for y.

But the z functions, one of the indeces which is zero, is taken directly from the table. The function of z, both of the lower indeces which in essence are units, are derived by formula (21).

The functions of z, differing only in the order of lower indeces, i.e. $Z^{(n)}_{\alpha,\beta}$ and $Z^{(n)}_{\beta\alpha}$ ($\alpha + \beta = k$) are obtained by a simple substitution x for y or conversely as the case may be.

Therefore, only functions remaining are those requiring calculation by formulas (22) or (23). There are the results:

For n = 1 and n = 2 the corresponding functions of z are immediately expressed. For n = 3, only $Z^{(3)}_{2,1} = \frac{1}{2} x_1^2 \; y_1$, should be written, while all others are immediately expressed.

For n = 4, only $Z^{(4)}_{2,1} = x_1 x_2 y_1 + \frac{1}{2} x_1^2 \; y_2$, and $Z^{(4)}_{3,1} = \frac{1}{6} x_1^3 y_1$ и $Z^{(4)}_{2,2} = \frac{1}{4} x_1^2 y_1^2$, is discovered by calculation, while the rest is immediately expressed.

For n = 5, the following functions are found by calculation:

$$Z_{2,1}^{(5)} = \frac{1}{2} x_1^2 y_3 + x_1 x_2 y_2 + x_1 x_3 y_1 + \frac{1}{2} x_2^2 y_1; \quad Z_{3,1}^{(5)} = \frac{1}{2} x_1^2 x_2 y_1 + \frac{1}{6} x_1^3 y_2;$$

$$Z_{2,2}^{(5)} = \frac{1}{2} x_1 x_2 y_1^2 + \frac{1}{2} x_1^2 y_1 y_2; \quad Z_{4,1}^{(5)} = \frac{1}{24} x_1^4 y_1; \quad Z_{3,2}^{(5)} = \frac{1}{12} x_1^3 y_1^2,$$

while the remaining functions are directly expressed.

Finally, for n = 6, we obtain by calculation:

$$Z_{2,1}^{(6)} = \frac{1}{2} x_1^2 y_4 + x_1 x_2 y_3 + x_1 x_3 y_2 + x_1 x_4 y_1 + x_2 x_3 y_1 + \\ + \frac{1}{2} x_2^2 y_2;$$

$$Z_{3,1}^{(6)} = \frac{1}{6} x_1^3 y_3 + \frac{1}{2} x_1^2 x_2 y_2 + \frac{1}{2} x_1 x_2^2 y_1 + \frac{1}{2} x_1^2 x_3 y_1;$$

$$Z_{2,2}^{(6)} = \frac{1}{4} x_1^2 y_2^2 + \frac{1}{2} x_1^2 y_1 y_3 + x_1 x_2 y_1 y_2 + \frac{1}{2} x_1 x_3 y_1^2 + \frac{1}{4} x_2^2 y_1^2;$$

$$Z_{4,1}^{(6)} = \frac{1}{6} x_1^3 x_2 y_1 + \frac{1}{24} x_1^4 y_2;$$

$$Z_{3,2}^{(6)} = \frac{1}{6} x_1^3 y_1 y_2 + \frac{1}{4} x_1^2 x_2 y_1^2;$$

$$Z_{5,1}^{(6)} = \frac{1}{120} x_1^5 y_1; \quad Z_{4,2}^{(6)} = \frac{1}{48} x_1^4 y_1^2;$$

$$Z_{3,3}^{(6)} = \frac{1}{36} x_1^3 y_1^3,$$

while the remaining functions are directly expressed.

If we turn our attention to an instance of complex functions z = f(x, y, u) with these intermediate arguments, then for its complete compilation of nth order derivative, it becomes necessary to express functions of z with three lower indeces by a formula similar to (40). $Z_{\alpha,\beta,\gamma}^{(n)} = \Sigma X_\alpha^{(n_1)} Y_\beta^{(n_2)} U_\gamma^{(n_3)},$

where $\alpha + \beta + \gamma = k, \; n_1 + n_2 + n_3 = n, \; n_1 \geqslant \alpha, \; n_2 \geqslant \beta, \; n_3 \geqslant \gamma.$

Evidently, only functions with lower values not = 0 are subject to calculation, since the latter are expressed directly from the preceeding tables. Further, from functions

$$Z_{\alpha,\beta,\gamma}^{(n)}, \; Z_{\beta,\alpha,\gamma}^{(n)}, \; Z_{\gamma,\beta,\alpha}^{(n)}, \; Z_{\alpha,\gamma,\beta}^{(n)}, \; Z_{\beta,\gamma,\alpha}^{(n)}, \; Z_{\gamma,\alpha,\beta}^{(n)},$$

it is evidently sufficient to express one of these, since the rest are obtained by a simple transformation of the letters.

As a result of these calculations, the following functions are obtained:

for для $n=3$; $Z^{(3)}_{1,1,1} = x_1y_1u_1$;

for для $n=4$; $Z^{(4)}_{1,1,1} = x_1y_1u_2 + x_1y_2u_1 + x_2y_1u_1$;

$$Z^{(4)}_{2,1,1} = \frac{1}{2} x_1^2 y_1 u_1.$$

for для $n=5$: $Z^{(5)}_{1,1,1} = x_1y_1u_3 + x_1y_2u_2 + x_1y_3u_1 + x_2y_1u_2 + x_2y_2u_1 + x_3y_1u_1$

$$Z^{(5)}_{2,1,1} = \frac{1}{2} x_1^2 y_1 u_2 + \frac{1}{2} x_1^2 y_2 u_1 + x_1 x_2 y_1 u_1;$$

$$Z^{(5)}_{3,1,1} = \frac{1}{6} x_1^3 y_1 u_1;$$

$$Z^{(5)}_{2,2,1} = \frac{1}{4} x_1^2 y_1^2 u_1.$$

for для $n=6$: $Z^{(6)}_{1,1,1} = x_1y_1u_4 = x_1y_3u_3 + x_1y_3u_2 + x_1y_4u_1 + x_2y_1u_3 + x_2y_2u_2 + x_2y_3u_1 + x_3y_1u_2 + x_3y_2u_1 + x_4y_1u_1$;

$$Z^{(6)}_{2,1,1} = \frac{1}{2} x_1^2 y_1 u_3 + \frac{1}{2} x_1^2 y_2 u_2 + \frac{1}{2} x_1^2 y_3 u_1 + x_1 x_2 y_1 u_2 + x_1 x_2 y_2 u_1 + x_1 x_3 y_1 u_1 + \frac{1}{2} x_2^2 y_1 u_1;$$

$$Z^{(6)}_{3,1,1} = \frac{1}{6} x_1^3 y_1 u_2 + \frac{1}{6} x_1^3 y_2 u_1 + \frac{1}{2} x_1^2 x_2 y_1 u_1;$$

$$Z^{(6)}_{2,2,1} = \frac{1}{4} x_1^2 y_1^2 u_2 + \frac{1}{2} x_1 x_2 y_1^2 u_1 + \frac{1}{2} x_1^2 y_1 y_2 u_1;$$

$$Z^{(6)}_{4,1,1} = \frac{1}{24} x_1^4 y_1 u_1;$$

$$Z^{(6)}_{3,2,1} = \frac{1}{12} x_1^3 y_1^2 u_1.$$

We limit ourselves with these figures.

These formulas demonstrate that the calculation operation of higher derivatives from complex functions with any number of intermediate arguments leads to a simple expression of necessary quantities from the tables, which may be pre-compiled and which may be useful in all cases that could be encountered in the introductions.

Similar operations utilized by us in another project was devoted to the computation of orders determining the movement of Saturn's* satellites.

*Concerning orders determining the movement of Saturn's satellites with the calculation of the Sun's perturbation effect, Bulletin of the Moscow University No. 6, 1949.